

Nova metoda za ocjenu stabilnosti automobilskih nelinearnih sustava upravljanja

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UNIVERSITY OF ZAGREB
FACULTY OF ELECTRICAL ENGINEERING AND COMPUTING

MASTER THESIS No. 48

**A NOVEL METHOD FOR ASSESSING STABILITY IN
NONLINEAR AUTOMOTIVE CONTROL SYSTEMS**

Teo Banov

Zagreb, February 2024

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Student: **Teo Banov (0069075129)**
Study: Information and Communication Technology
Profile: Control Systems and Robotics
Mentor: asst. prof. Branimir Novoselnik

Title: **A novel method for assessing stability in nonlinear automotive control systems**

Description:

Methods for assessing the stability of non-linear dynamic systems are relatively limited, especially in terms of industrial application. The ever-increasing number of active subsystems and the complexity of their interconnection in land vehicles is one of the main drivers of safety progress in the automotive sector over the past 50 years. However, processes for methodical analysis of stability of control systems in land vehicles are not sufficiently developed despite their great importance for further improvement of active safety systems and efficient implementation of active safety measures with minimal use of sub-optimal, empirically defined and too conservative methods based on redundancy layers that increase costs and threaten the achievement of the full performance potential of active systems. The task of the graduate thesis is to respond to these requirements by defining a new framework that will enable an objective assessment of the stability of car steering systems in cases where the dynamic system is close to the edge of its limitations and has strong non-linearities. Furthermore, specific provisions should be defined to extend this framework to the applicable and robust method that original equipment manufacturers (OEM) and TIER 1 automotive component manufacturers can use in the design verification phases through the use of high-fidelity simulators as well as testing of the vehicles themselves. The research will be conducted in cooperation with Bugatti-Rimac using specialized driver-in-the-loop and hardware-in-the-loop simulators as well as test vehicles to validate the developed framework.

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Zadatak: **Nova metoda za ocjenu stabilnosti automobilskih nelinearnih sustava upravljanja**

Opis zadatka:

Metode za procjenu stabilnosti nelinearnih dinamičkih sustava relativno su ograničene, posebno u smislu industrijske primjene. Stalno rastući broj aktivnih podsustava te kompleksnost njihove međusobne sprege u kopnenim vozilima jedan je od glavnih motivatora sigurnosnog napretka u automobilskom sektoru u proteklih 50 godina. Međutim, procesi za metodičku analizu stabilnosti upravljačkih sustava u kopnenim vozilima nisu dovoljno razvijeni unatoč njihovoj velikoj važnosti za daljnje poboljšanje sustava aktivne sigurnosti te efikasnu implementaciju mjera aktivne sigurnosti uz minimalno korištenje neoptimalnih, empirijski definiranih te suviše konzervativnih metoda temeljenih na slojevima zalihosti koje povećavaju troškove i ugrožavaju postizanje punog potencijala performansi aktivnih sustava. Zadatak diplomskog rada je odgovoriti na navedene zahtjeve kroz definiranje novog okvira koji će omogućiti objektivnu procjenu stabilnosti upravljačkih sustava automobila u slučajevima kada se dinamički sustav nalazi blizu ruba svojih ograničenja te ima izražene jake nelinearnosti. Nadalje, potrebno je definirati konkretne odredbe za proširenje navedenog okvira na primjenjivu i robusnu metodu koju mogu koristiti proizvođači originalne opreme (OEM) i TIER 1 proizvođači automobilskih komponenti u fazama provjere koncepta i verifikacije dizajna kroz korištenje simulatora visoke vjernosti kao i testiranja samih vozila. Istraživanje će biti provedeno u suradnji s Bugatti-Rimcem uz korištenje specijaliziranih Driver-in-the-Loop i Hardware-in-the-loop simulatora kao i testnih vozila za provjeru valjanosti razvijenog okvira.

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1. Introduction

Vehicle safety and performance in dynamic environments has been a great challenge in automotive engineering. As automotive technologies evolve, so does the need for robust methodologies to analyze and ensure the stability of vehicle dynamics, more specifically lateral dynamics. This research focuses on implementing two fundamental techniques for stability analysis: phase plane analysis and Lyapunov analysis. Phase Plane Analysis provides a visual approach to studying the behavior of dynamical systems. By plotting state variables in a phase plane, interesting patterns of system behavior become visually apparent. This method is particularly valuable in analyzing the qualitative behavior of nonlinear systems, offering insights into stability, limit cycles, and other nonlinear behaviour. Lyapunov Analysis offers a rigorous mathematical framework for assessing stability in dynamical systems. Named after Aleksandr Lyapunov, this method relies on the examination of Lyapunov functions to determine the stability properties of equilibrium points. Through Lyapunov's direct method, we can establish different types of stability of dynamical systems, providing a deeper understanding of their behavior.

2. Nonlinear Control Systems

Every physical system in real world is nonlinear. Its dynamics are described as set of differential equations. Nonlinearities can be result of energy limitations, imperfections of the system, have intentionally added nonlinear elements(e.g. hysteresis) or simply have dynamics that behave in a nonlinear way. Unlike linear system, nonlinear systems can have multiple equilibrium points, limit cycles, chaos and many other phenomena.

As previously mentioned, dynamic system can be represented by a set of nonlinear differential equations:

$$\dot{\mathbf{x}} = f(\mathbf{x}, t). \quad (2.1)$$

If there exists some control law $\mathbf{u} = g(\mathbf{x}, t)$, then closed loop dynamics are:

$$\dot{\mathbf{x}} = f(\mathbf{x}, g(\mathbf{x}, t), t). \quad (2.2)$$

Theory of nonlinear control has the following problems:

1. Analysis problem: search for properties of the system, how it behaves under different conditions and its stability.
2. Synthesis problem: finding parameters, structure and elements of the system required to create and develop controllers or algorithms to achieve desired performance.

Analysis problem will be covered in greater detail in later chapters while synthesis will not.

2.1. Phase Trajectories of Nonlinear Systems

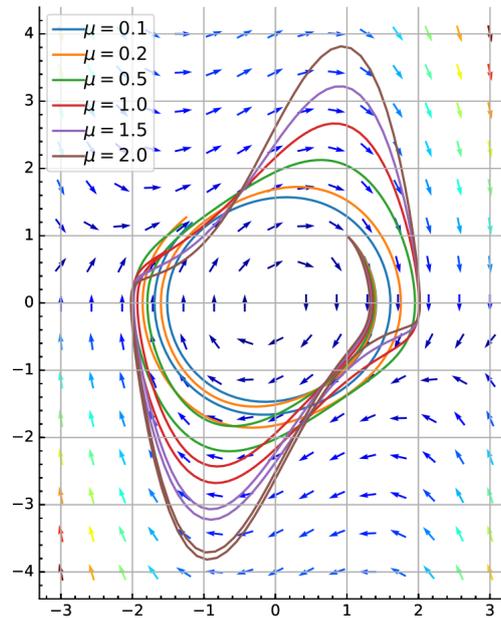


Figure 2.1. Phase portrait of Van der Pol's equation, source:[7]

Best way to develop intuition how a system behaves is to visualize it. Very powerful tool to achieve that is Phase portrait. One of its limitations is number of states that can be plotted, up to three. There are multiple methods such as analytical method and method of isoclines. In this work, phase plane construction will be based on solving systems of equations using Matlabs solvers. There are multiple different types of equilibrium points which is shown on figure 2.2.

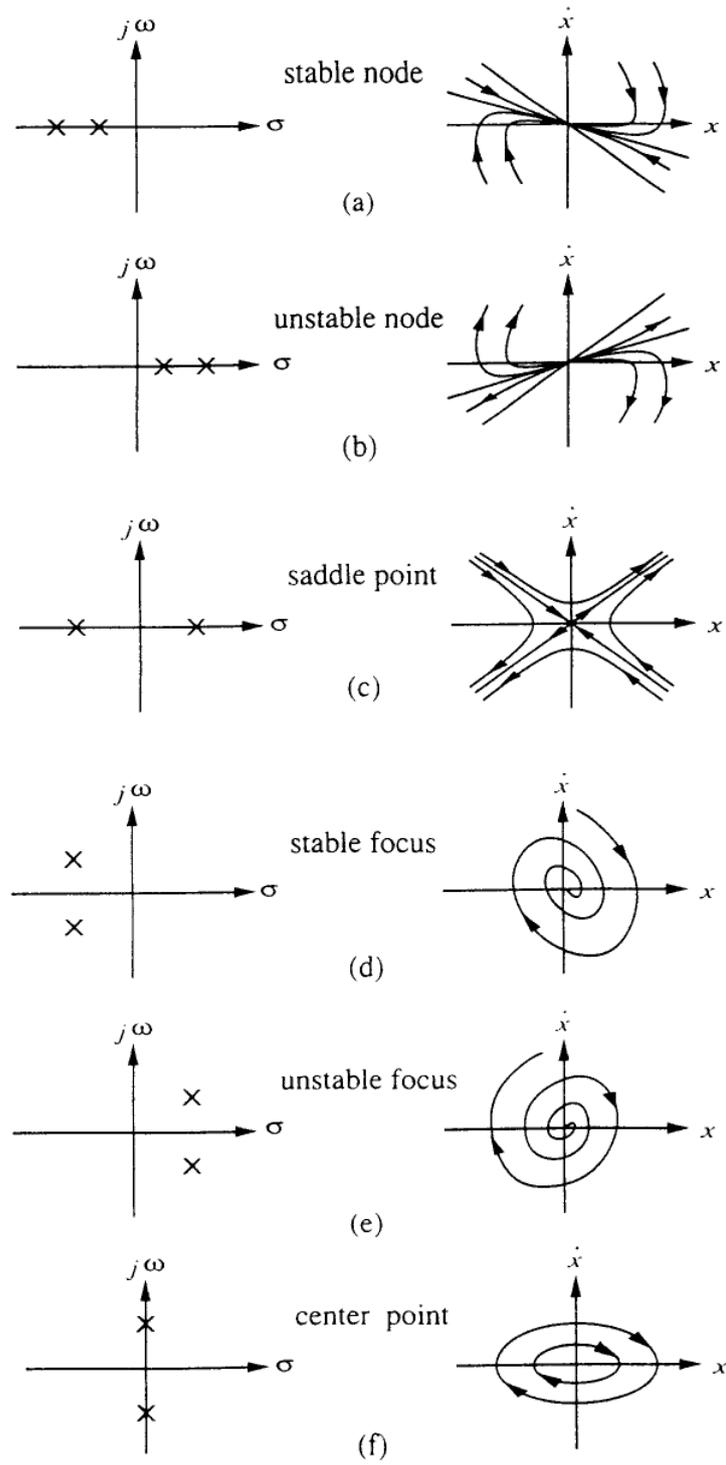


Figure 2.2. Types of equilibrium points source:[1]

2.2. Lyapunov Stability

To define stability of a system, we need to first describe an autonomous system. It is a system which does not change in time (time-invariant) and has no outside inputs that affect the system.

Definition 2.1. *Nonlinear system is autonomous if it can be described with a system of first-order differential equations*

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}) \quad (2.3)$$

where \mathbf{f} is $n \times 1$ vector function and \mathbf{x} is $n \times 1$ state vector and n is number of states, otherwise it is non-autonomous.

Next we define equilibrium points.

Definition 2.2. *A state \mathbf{x}_e is a state of equilibrium if trajectory of the system equals $\mathbf{x}(t) = \mathbf{x}_e$ and remains in that state for all future t .*

State vector being constant satisfies equation

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}_e) = 0. \quad (2.4)$$

Let us describe concept of Lyapunov stability. We will assume that equilibrium point is in the origin, $\mathbf{x}_e = 0$. According to [1], Lyapunov stability is introduced by the following definition.

Definition 2.3. *The equilibrium state $\mathbf{x}_e = 0$ is stable if, for any $R > 0$, there exists $r > 0$, such that if $\|\mathbf{x}(0)\| < r$, then $\|\mathbf{x}(t)\| < R$ for all $t \geq 0$. Otherwise, the equilibrium point is unstable.*

More intuitively, system is stable in Lyapunov sense if initial state $\mathbf{x}(0) = \mathbf{x}_0$ is inside the ball (or hypersphere if there are more than 3 states) of radius r and the trajectory from that initial state is always inside ball (hypersphere) of radius R .

Often, this definition of stability is not enough. In some cases, we want system to reach equilibrium or even to reach it fast (in finite time). Now we will expand concept of stability by introducing asymptotic and exponential stability.

Definition 2.4. *Equilibrium state $\mathbf{x}_e = 0$ is asymptotically stable if it is stable and there exist some $r > 0$ such that $\|\mathbf{x}(0)\| < r$ implies $\mathbf{x}(t) \rightarrow 0$ as $t \rightarrow \infty$.*

To put simply, system is stable and state converges to equilibrium as time goes to ∞ .

Exponential stability not only makes sure that system is asymptotically stable, but how fast it converges to equilibrium as well.

Definition 2.5. *Equilibrium state $\mathbf{x}_e = 0$ is exponentially stable if it is stable and there exist numbers $\alpha, \lambda \in \mathbb{R}_+$ such that*

$$\forall t, \|\mathbf{x}(t)\| \leq \alpha \|\mathbf{x}(0)\| e^{-\lambda t}. \quad (2.5)$$

Simply, this means that trajectory converges to equilibrium faster than exponential function $\alpha \|\mathbf{x}(0)\| e^{-\lambda t}$.

In above definitions of stability, we considered local stability considering initial states within some hypersphere of radius r . In terms of global stability, system will be stable, asymptotically or exponentially stable for any initial state.

2.3. Lyapunov's Direct Method

There are two methods to Lyapunov stability analysis:

1. linear model analysis
2. direct analysis.

Linear model analysis, as the name suggests, is based on linearization of system and its trajectory is often visualised with 2-state systems. Analysis of linearized system is good only in small region which will be shown why. Another problem is that solving nonlinear differential equation can be extremely difficult for complex systems. Basic idea of direct method is that we can analyze stability of a system without solving differential equation. That is done through use of Lyapunov function $V(x)$. It is a scalar function that originally started as mathematical extension of total energy of a system. If energy is constantly dissipated, then the system will eventually end up in equilibrium point.

2.3.1. Properties of Lyapunov Functions

Since energy can only be positive, Lyapunov function needs to be positive as well.

Definition 2.6. A scalar continuous function $V(\mathbf{x})$ is positive definite if $V(0) = 0$ and for $\mathbf{x} \neq 0$, $V(\mathbf{x}) > 0$

Another property of Lyapunov function is that it is monotone and decreasing as \mathbf{x} and $\dot{\mathbf{x}}$ vary. It is described with \dot{V} , a derivative of V along system trajectory.

Definition 2.7. For some system, if $V(\mathbf{x})$ is positive definite, has continuous partial derivatives and time derivative along any trajectory is negative semi-definite $\dot{V}(\mathbf{x}) \leq 0$, then $V(\mathbf{x})$ is Lyapunov function for that system.

2.3.2. Local and Global Stability

System can be stable inside hypersphere of radius R , but unstable outside (or we do not care about stability outside of it). This brings us to important concept: local and global stability.

Definition 2.8. *If there exists $V(\mathbf{x})$, with continuous partial first derivatives, inside hypersphere with radius R such that $V(\mathbf{x})$ is positive definite and $\dot{V}(\mathbf{x})$ is negative semi-definite, then the system is stable. If $\dot{V}(\mathbf{x})$ is negative definite, the system is asymptotically stable. Proof is in [1].*

For global stability, hypersphere needs to be extended to whole state space, in case of physical system, \mathbb{R}^n . Another extension needs to be made: $V(\mathbf{x})$ has to be radially unbounded, meaning that $V(\mathbf{x}) \rightarrow \infty$ as $\|\mathbf{x}\| \rightarrow \infty$. Everything combined gives the following definition.

Definition 2.9. *Let $V(\mathbf{x})$ be Lyapunov function of some system, with continuous first order derivations. If the following is valid:*

1. $V(\mathbf{x})$ is positive definite
2. $\dot{V}(\mathbf{x})$ is negative definite
3. $V(\mathbf{x}) \rightarrow \infty$ as $\|\mathbf{x}\| \rightarrow \infty$,

then equilibrium is globally asymptotically stable.

2.4. Methods forming Lyapunov functions

One of the problems of finding Lyapunov function is that there are no general ways to find one for nonlinear system as each system behaves completely differently.

2.4.1. Linear method

For a linear system described by $\dot{\mathbf{x}}(t) = A\mathbf{x}(t)$ one of the Lyapunov functions is $V(\mathbf{x}) = \mathbf{x}^T P \mathbf{x}$. Matrix P is positive definite and therefore scalar function $V(\mathbf{x})$ will be positive definite as well. Last property that needs to be satisfied is negative definiteness of the gradient:

$$\begin{aligned}\dot{V}(\mathbf{x}) &= \dot{\mathbf{x}}^T P \dot{\mathbf{x}} + \mathbf{x}^T P \dot{\mathbf{x}} = \mathbf{x}^T A^T P \mathbf{x} + \mathbf{x}^T P A \mathbf{x} \\ \dot{V}(\mathbf{x}) &= \mathbf{x}^T (A^T P + P A) \mathbf{x} = -\mathbf{x}^T Q \mathbf{x} < 0.\end{aligned}\tag{2.6}$$

Given 2.6, linear system will be asymptotically stable at equilibrium point if:

$$A^T P + P A = -Q,\tag{2.7}$$

where Q can be identity matrix[2].

Nonlinear system with n equations and states can be linearized by determining its Jacobian:

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \dots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}.\tag{2.8}$$

This can give great insight into stability of a nonlinear system.

3. Vehicle model

In previous chapter, brief introduction to nonlinear system analysis was given. The goal is to perform an analysis on lateral dynamics of a vehicle and get an insight into its stability under different conditions. To do that, first we need to define the model. Vehicle model that will be used is simplified bicycle model. For lateral dynamics, we will derive the 2-degree of freedom bicycle model $f(v_y, \dot{\psi})$ with lateral speed v_y and yaw rate $\dot{\psi}$ as system states and then expand it to 4-DOF model by including front and rear tyre relaxation lengths.

3.1. Lateral dynamics of bicycle model

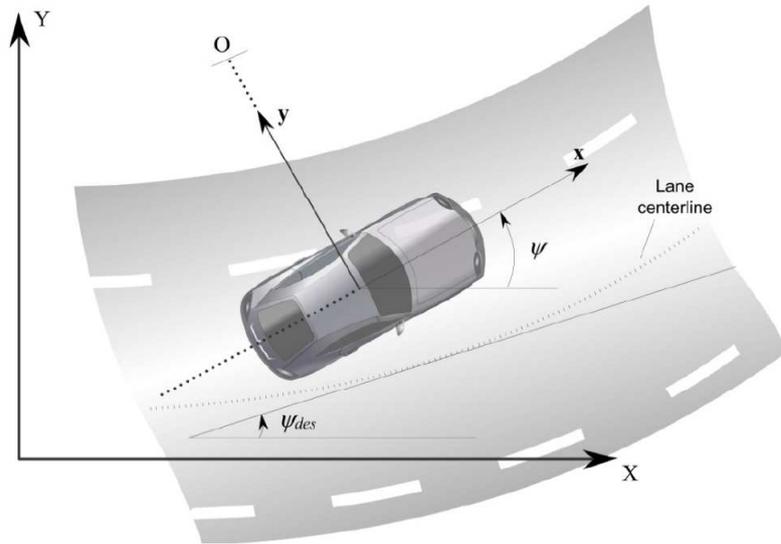


Figure 3.1. Dynamic bicycle model source:[4]

First, we start with defining lateral position y which is measured from vehicles centered of gravity to center of rotation of the vehicle, yaw angle ψ is measured with respect to global coordinate X-axis and lastly vehicle speed, longitudinal speed denoted by v_x as shown on figure 3.1. We now apply Newton's second law for motion for y-axis of vehicles local coordinate frame:

$$ma_y = F_{yf} + F_{yr},$$

where Front and rear lateral tyre forces, also shown in figure 3.2, are F_{yf} and F_{yr} and a_y is inertial acceleration along local y-axis, m is mass of the vehicle. Furthermore, a_y consists of two components:

$$a_y = \frac{\partial^2 y}{\partial t^2} + v_x \frac{\partial \psi}{\partial t}, \quad (3.1)$$

where first component is lateral acceleration as a result of motion along y-axis and second component is centripetal acceleration due to rotation around center of gravity. Substituting 3.1. into 3.1. gives:

$$m \left(\frac{\partial^2 y}{\partial t^2} + v_x \frac{\partial \psi}{\partial t} \right) = F_{yf} + F_{yr} \quad (3.2)$$

$$m \frac{\partial^2 y}{\partial t^2} = F_{yf} + F_{yr} - m v_x \frac{\partial \psi}{\partial t}.$$

Newton's second law for rotation of the vehicle around z-axis in local coordinate frame is:

$$J_z \frac{\partial^2 \psi}{\partial t^2} = l_f F_{yf} - l_r F_{yr} \quad (3.3)$$

where l_f is distance from center of gravity (cog) to front wheel, and l_r distance from cog to rear wheel, J_z is rotational inertia of z-axis. Now the lateral model can be extended to include steering angle of front wheel δ_f as shown in figure 3.2:

$$m \dot{v}_y = F_{yf} \cos(\delta_f) + F_{yr} - m v_x \dot{\psi} \quad (3.4)$$

$$J_z \ddot{\psi} = l_f F_{yf} \cos(\delta_f) - l_r F_{yr}.$$

3.2. Wheel geometry

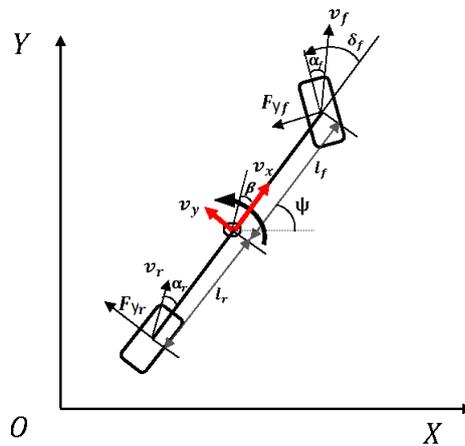


Figure 3.2. Dynamic bicycle model source:[3]

Lateral forces depend on the slip angle of the tyres $F_y(\alpha)$. Slip angle of a tyre is the angle between orientation of the tyre and velocity vector of the tyre. This relation is shown on figure 3.2.

For the front wheel, slip angle is given by:

$$\alpha_f = \arctan \frac{v_{f2}}{v_{f1}},$$

where it can be seen that

$$v_{f1} = v_{fx} \cos(\delta_f) + v_{fy} \sin(\delta_f) \quad (3.5)$$

$$v_{f2} = -v_{fx} \sin(\delta_f) + v_{fy} \cos(\delta_f).$$

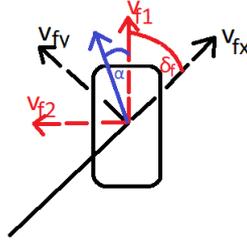


Figure 3.3. Front wheel geometry

Looking at figure 3.3, we can see that $v_{fx} = v_x$ and $v_{fy} = v_y + \dot{\psi}l_f$ and by substituting into 3.5 gives:

$$\alpha_f = \arctan \frac{-v_x \sin \delta_f + (v_y + \dot{\psi}l_f) \cos \delta_f}{v_x \cos \delta_f + (v_y + \dot{\psi}l_f) \sin \delta_f}. \quad (3.6)$$

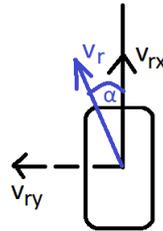


Figure 3.4. Rear wheel geometry

For the rear wheel, slip angle is:

$$\alpha_r = \arctan \frac{v_{ry}}{v_{rx}},$$

from figure 3.4 we see $v_{ry} = v_y - \dot{\psi}l_r$ and $v_x = v_{rx}$. Again, by substitution, we get rear wheel slip angle:

$$\alpha_r = \arctan \frac{v_y - \dot{\psi}l_r}{v_x}. \quad (3.7)$$

3.3. Tyre Model

Next step in deriving lateral model is to select tyre model. There are many tyre models such as Pacejka's magic formula, Dugoff's model, TMeasy and others. Pacejka's magic formula will be used because it is widely used in automobile industry. It is an empirical model and the reason

behind wide use is that analytical models are not accurate for larger slip angles. Magic formula follows[4]:

$$F_y = D \sin \{C \arctan [B\alpha - E (B\alpha - \arctan (B\alpha))]\}, \quad (3.8)$$

where F_y is lateral force, α is slip angle and

- B is stiffness factor
- C is shape factor
- D is maximum value
- E is curvature factor.

In our model, lateral force has negative value:

$$F_y = -D \sin \{C \arctan [B\alpha - E (B\alpha - \arctan (B\alpha))]\}. \quad (3.9)$$

Due to sin and arctan being odd functions, sign can be moved to equations for slip angles which gives following equations for front and rear lateral forces and slip angles:

$$F_{yf} = D_f \sin \{C_f \arctan [B_f \alpha_f - E_f (B_f \alpha_f - \arctan (B_f \alpha_f))]\} \quad (3.10)$$

$$F_{yr} = D_r \sin \{C_r \arctan [B_r \alpha_r - E_r (B_r \alpha_r - \arctan (B_r \alpha_r))]\} \quad (3.11)$$

$$\alpha_f = \arctan \frac{v_x \sin \delta_f - (v_y + \dot{\psi} l_f) \cos \delta_f}{v_x \cos \delta_f + (v_y + \dot{\psi} l_f) \sin \delta_f} \quad (3.12)$$

$$\alpha_r = \arctan \frac{-v_y + \dot{\psi} l_r}{v_x} \quad (3.13)$$

3.4. Extension of Lateral Model

One of the phenomena of tyres that can have big impact on behaviour and stability of a vehicle is tyre relaxation length σ . It is related to distance that tyre travels for lateral force to reach certain percentage of steady-state after step change of slip angle. Since this phenomenon models delay in side slip α , it is modeled with first order linear differential equation[5]:

$$\frac{\sigma}{v_x} \frac{\partial \alpha_1}{\partial t} + \alpha_1 = \alpha. \quad (3.14)$$

It can be noticed that this phenomenon behaves as a low pass filter

$$\frac{\alpha_1(s)}{\alpha(s)} = \frac{\frac{v_x}{\sigma}}{s + \frac{v_x}{\sigma}} \quad (3.15)$$

and as such will be modeled in Simulink.

By substituting 3.12 and 3.13 into 3.14 we get the model for front and rear tyre relaxation:

$$\begin{aligned}\dot{\alpha}_{1f} &= \frac{v_x}{\sigma_f} \left(-\alpha_{1f} + \arctan \frac{v_x \sin \delta_f - (v_y + \dot{\psi} l_f) \cos \delta_f}{v_x \cos \delta_f + (v_y + \dot{\psi} l_f) \sin \delta_f} \right) \\ \dot{\alpha}_{1r} &= \frac{v_x}{\sigma_r} \left(-\alpha_{1r} + \arctan \frac{-v_y + \dot{\psi} l_r}{v_x} \right)\end{aligned}\tag{3.16}$$

To include tyre relaxation into our model, lateral tyre forces are:

$$\begin{aligned}F_{yf} &= D_f \sin \{C_f \arctan [B_f \alpha_{1f} - E_f (B_f \alpha_{1f} - \arctan (B_f \alpha_{1f}))]\} \\ F_{yr} &= D_r \sin \{C_r \arctan [B_r \alpha_{1r} - E_r (B_r \alpha_{1r} - \arctan (B_r \alpha_{1r}))]\}\end{aligned}\tag{3.17}$$

4. Vehicle model Simulation

Nonlinear system stability analysis will be applied on a lateral bicycle model. Model of vehicle derived in previous chapter will be simulated

4.1. Simulink Model

Model of vehicle derived in previous chapter is created in Simulink, figure 4.1, full set of equation follows:

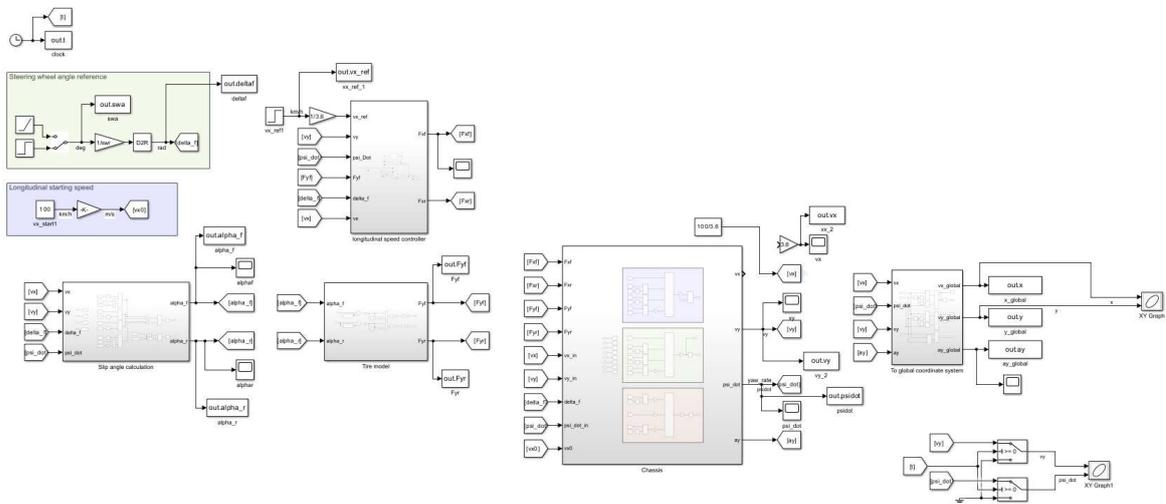


Figure 4.1. Simulink model of a bicycle

$$\begin{aligned}
 m\dot{v}_y &= F_{yf} \cos(\delta_f) + F_{yr} - mv_x\dot{\psi} \\
 J_z\ddot{\psi} &= l_f F_{yf} \cos(\delta_f) - l_r F_{yr} \\
 F_{yf} &= D_f \sin \{C_f \arctan [B_f \alpha_{1f} - E_f (B_f \alpha_{1f} - \arctan (B_f \alpha_{1f}))]\} \\
 F_{yr} &= D_r \sin \{C_r \arctan [B_r \alpha_{1r} - E_r (B_r \alpha_{1r} - \arctan (B_r \alpha_{1r}))]\} \\
 \dot{\alpha}_{1f} &= \frac{v_x}{\sigma_f} \left(-\alpha_{1f} + \arctan \frac{v_x \sin \delta_f - (v_y + \dot{\psi} l_f) \cos \delta_f}{v_x \cos \delta_f + (v_y + \dot{\psi} l_f) \sin \delta_f} \right) \\
 \dot{\alpha}_{1r} &= \frac{v_x}{\sigma_r} \left(-\alpha_{1r} + \arctan \frac{-v_y + \dot{\psi} l_r}{v_x} \right).
 \end{aligned} \tag{4.1}$$

Unfortunately, parameters of vehicle are classified and exact values will be omitted. Graphs will be shown with normalized data. Parameters of the vehicle are:

- m_f mass of front axle

- m_r mass of rear axle
- $m = m_f + m_r$ total mass of vehicle
- J_z inertia
- l wheelbase -distance between front and rear axles
- $l_r = \frac{l m_f}{m}$ distance between rear axle and center of gravity
- $l_f = l - l_r$ distance between front axle and center of gravity
- swr steering wheel ratio.

One difference with magic formula parameter D is that it is defined as $D = d * F_z$, where F_z is normal force of axle: F_{zf} for front axle and F_{zr} for rear axle as shown in figure 4.2. It is the maximum force that can the vehicle achieve lateraly:

$$\begin{aligned} F_{zf} &= \frac{l_r m g}{l} \\ F_{zr} &= \frac{l_f m g}{l}, \end{aligned} \quad (4.2)$$

where g is gravitational acceleration $g = 9.81 \frac{m}{s^2}$.

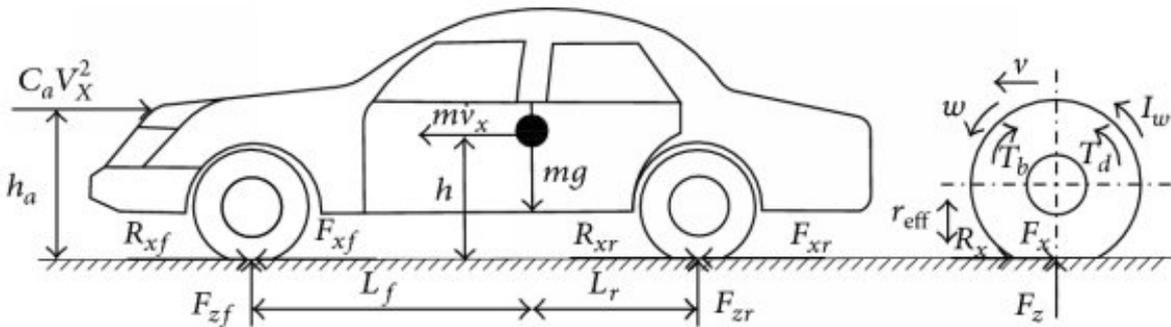


Figure 4.2. Longitudinal vehicle dynamics source:[6]

The model consists of multiple blocks. "Steering wheel reference" (figure 4.3) block takes steering wheel angle as input and outputs wheel angle δ_f based on steering wheel ratio swr . For stability analysis step input was used.

Since we only analyze lateral dynamics, longitudinal speed v_x is constant, its magnitude is defined in "Longitudinal starting speed" block.

Block "Slip angle calculation" implements equations 3.12 and 3.13 as shown in 4.4.

Implementation of front and rear tyre models in 3.17 is given in figure 4.5 with option to not use tyre relaxation.

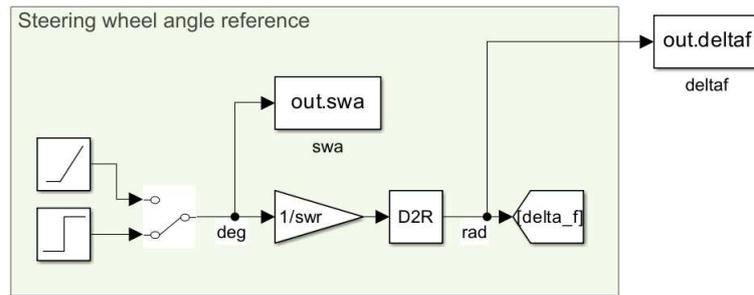


Figure 4.3. Steering wheel reference

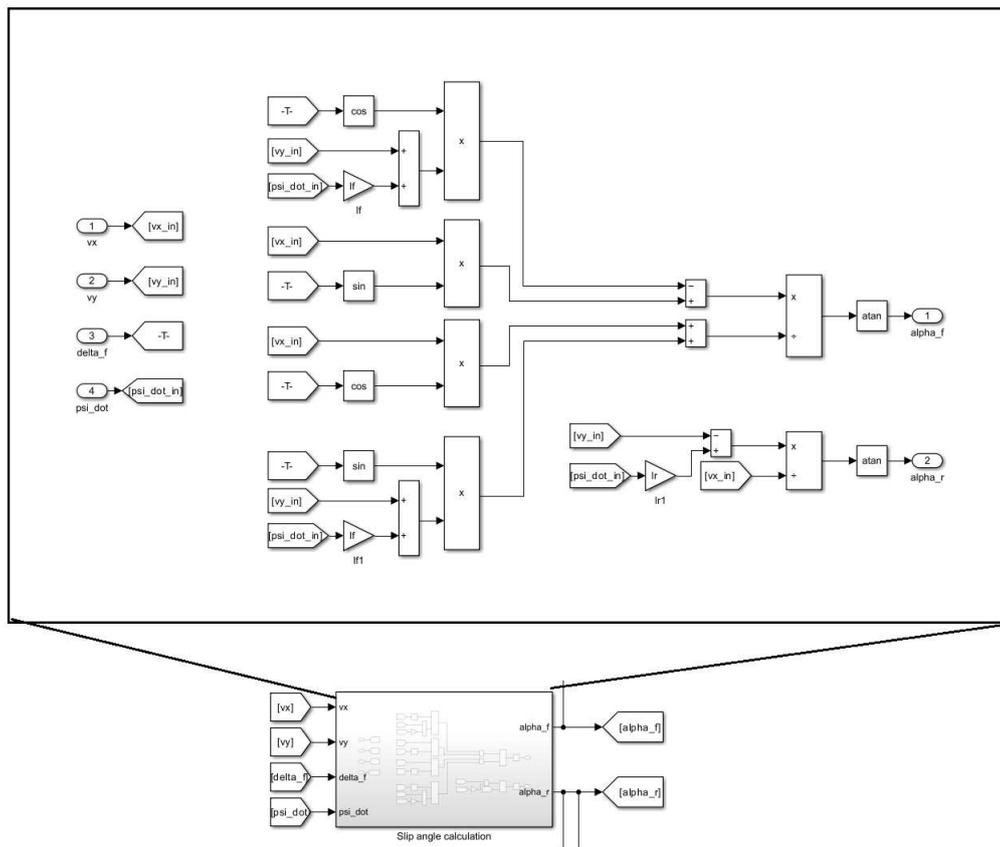


Figure 4.4. Slip angle

Although only lateral dynamics (3.4) are being considered , model in simulink has full dynamic model as shown in figure 4.6.

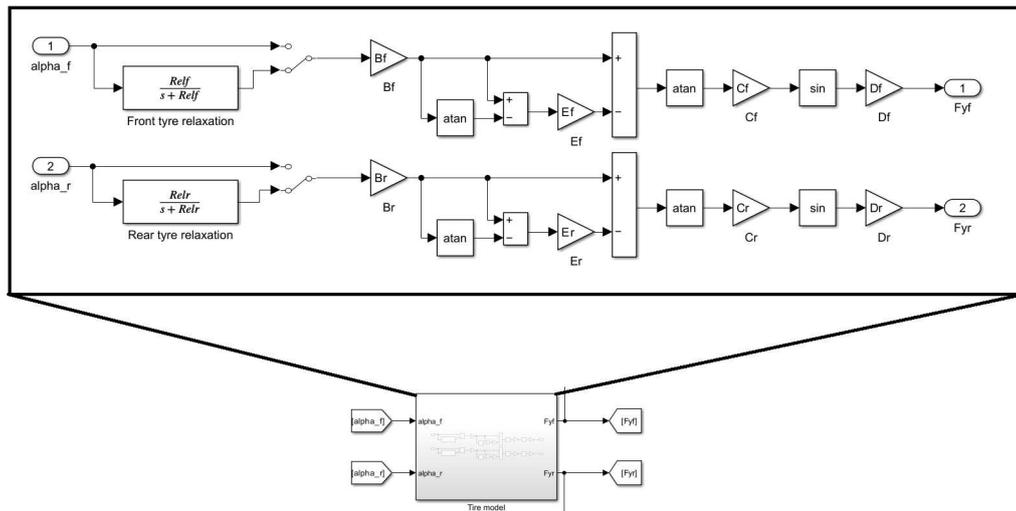


Figure 4.5. Tyre model

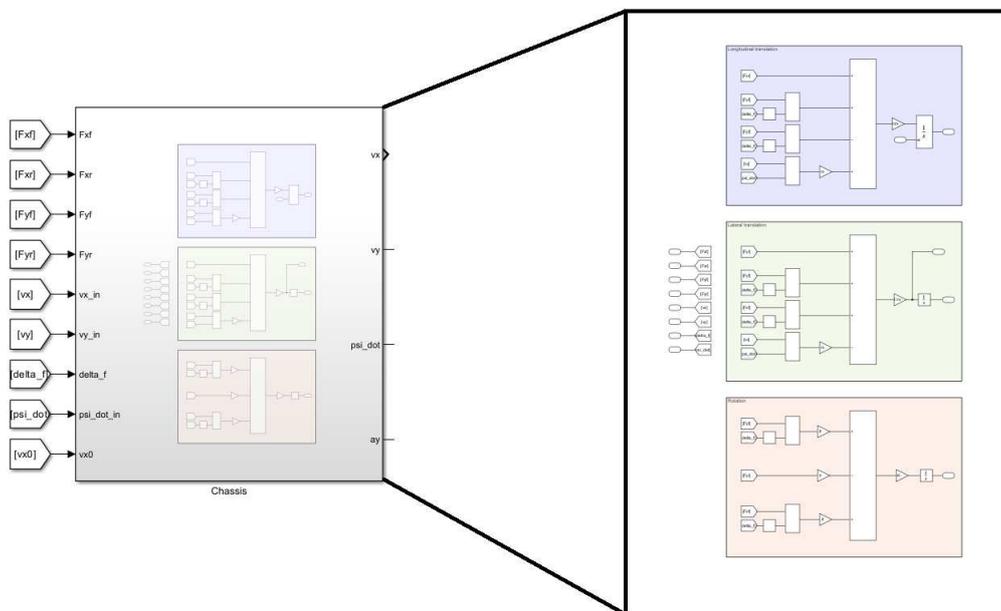


Figure 4.6. Lateral model

4.2. Simulation Results

Vehicle model is simulated using step inputs for steering wheel angles of $\delta_{strwhl} = 40^\circ$ and $\delta_{strwhl} = 70^\circ$.

In figure 4.7 can be seen that for step input $\delta_{strwhl} = 40^\circ$ system is stable while for $\delta_{strwhl} = 70^\circ$ is unstable. For unstable vehicle, both yaw rate and lateral speed increase in oscillations, which can also be seen in phase plane plot. Effect of those oscillations can also be seen on trajectory of the vehicle.

Vehicle was also simulated without tyre relaxation, in figure 4.8 can be seen that the vehicle is

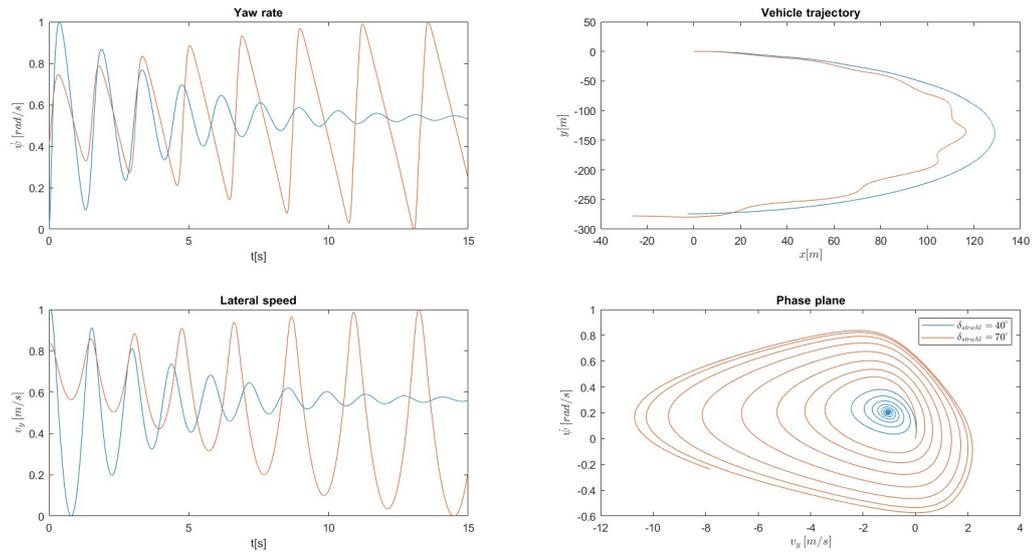


Figure 4.7. Simulation results with tyre relaxation

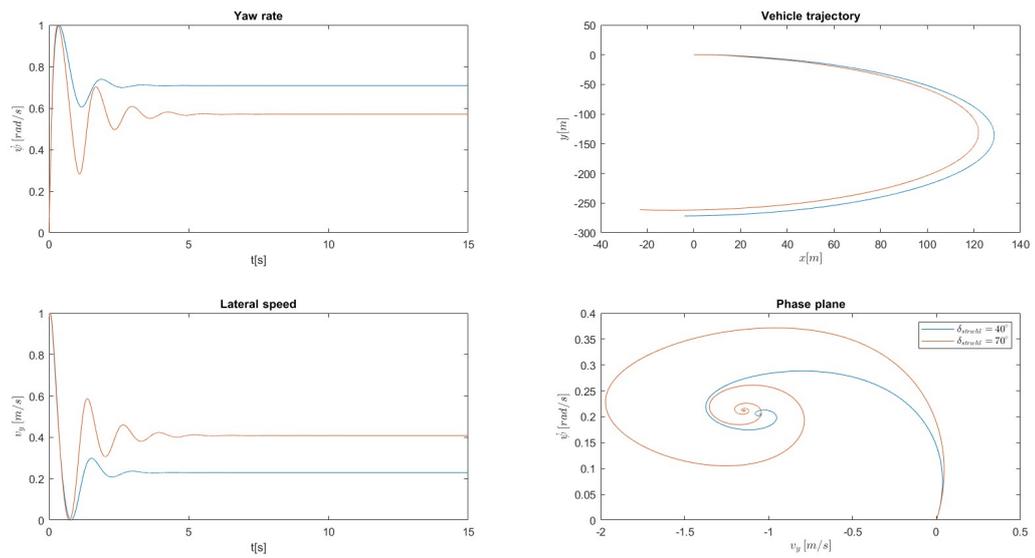


Figure 4.8. Simulation results without tyre relaxation

stable. Tyre relaxation can have big impact on vehicle stability as seen in simulations.

5. Stability analysis

In this chapter, stability of lateral vehicle model will be analyzed.

Equilibrium point of the vehicle model is determined by solving:

$$\begin{aligned}
 0 &= F_{yf} \cos(\delta_f) + F_{yr} - m v_x \dot{\psi} \\
 0 &= l_f F_{yf} \cos(\delta_f) - l_r F_{yr} \\
 0 &= \frac{v_x}{\sigma_f} \left(-\alpha_{1f} + \arctan \frac{v_x \sin \delta_f - (v_y + \dot{\psi} l_f) \cos \delta_f}{v_x \cos \delta_f + (v_y + \dot{\psi} l_f) \sin \delta_f} \right) \\
 0 &= \frac{v_x}{\sigma_r} \left(-\alpha_{1r} + \arctan \frac{-v_y + \dot{\psi} l_r}{v_x} \right).
 \end{aligned} \tag{5.1}$$

To find steering wheel angle for which vehicle becomes unstable, algorithm 5.1 was used. Lyapunov function $V(\mathbf{x}) = \mathbf{x}^T P \mathbf{x}$ is created for each wheel angle δ_f . Matrix P was found by solving 2.7 which satisfies Lyapunov property that is positive definiteness. Running the algorithm gives steering wheel angle of 54.0430° . Phase plane for given steering wheel angle is shown in figure 5.2. System is shown to be unstable if we simulation time is 15 seconds.

```

l = small  $\delta_f$ ;
r = large  $\delta_f$ ;
Mid = l+(r-l)/2;
while (r-l)/2>0.01
    mid = l+(r-l)/2;
    Find equilibrium point
    Find Jacobian of the system J
    Replace states of linearized system with equilibrium
    point in Jacobian matrix J
    P = sylvester(J,J,-eye(4));
    if eig(P)>0
        l = mid;
    else
        r = mid;
    end
End
stability_point = mid

```

Figure 5.1. Algorithm for finding unstable steering wheel angle

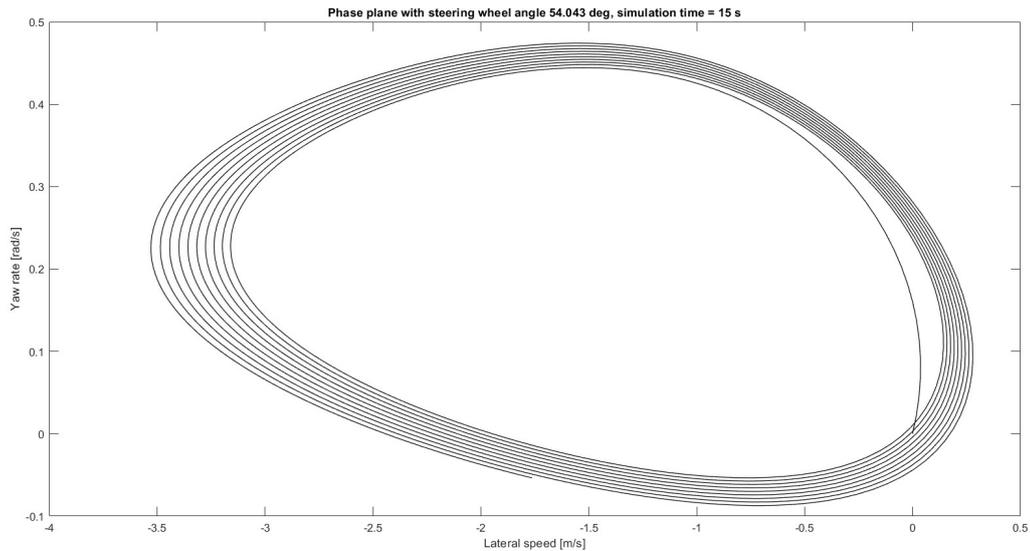


Figure 5.2. Phase plane for $\delta_{strwhl} = 54.043^\circ$ and $t = 15s$

Now we extend simulation time to 100 seconds. Figure 5.3 shows that the system is locally unstable only if we look at small simulation time. After extending it, we can see that equilibrium of the system is a stable limit cycle.

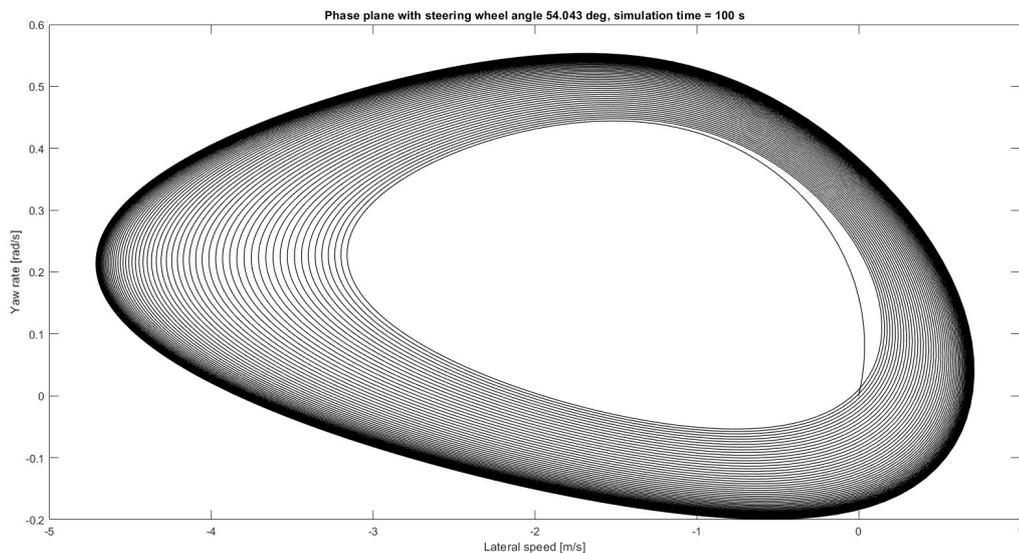


Figure 5.3. Phase plane for $\delta_{strwhl} = 54.043^\circ$ and $t = 100s$

Trajectories outside of the curve will tend to the curve while trajectories inside the curve will also tend to the curve. If a motion starts on the curve, it will stay on the curve. In the phase plane, a limit cycle is a closed curve - motion is periodic. It is also isolated - trajectories that are close to the curve will either converge or diverge from it. There are three kinds of limit cycles:

1. Stable limit cycles: all trajectories close to the curve converge to it

2. Unstable limit cycles: all trajectories close to the curve diverge to it
3. Semi-stable limit cycles: some trajectories converge, while others diverge.

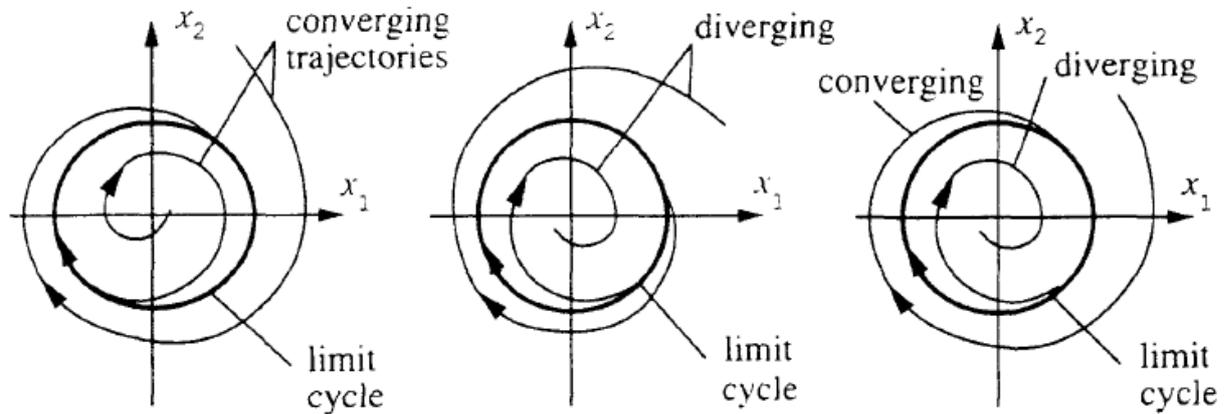


Figure 5.4. Left: stable limit cycle, middle: unstable limit cycle, right: semi-stable limit cycle:[1]

Asymptotic properties of limit cycles are described by following theorem:

Theorem 5.1 (Poincare-Bendixson). *If a trajectory of a system remains in finite region, then one of the following is true:*

1. *the trajectory goes to an equilibrium point*
2. *the trajectory tends to asymptotically stable limit cycle*
3. *trajectory itself is a limit cycle.*

Now we simulate our system from different starting points to see if our system really is stable limit cycle. Figure 5.5 confirms assumed type of equilibrium point.

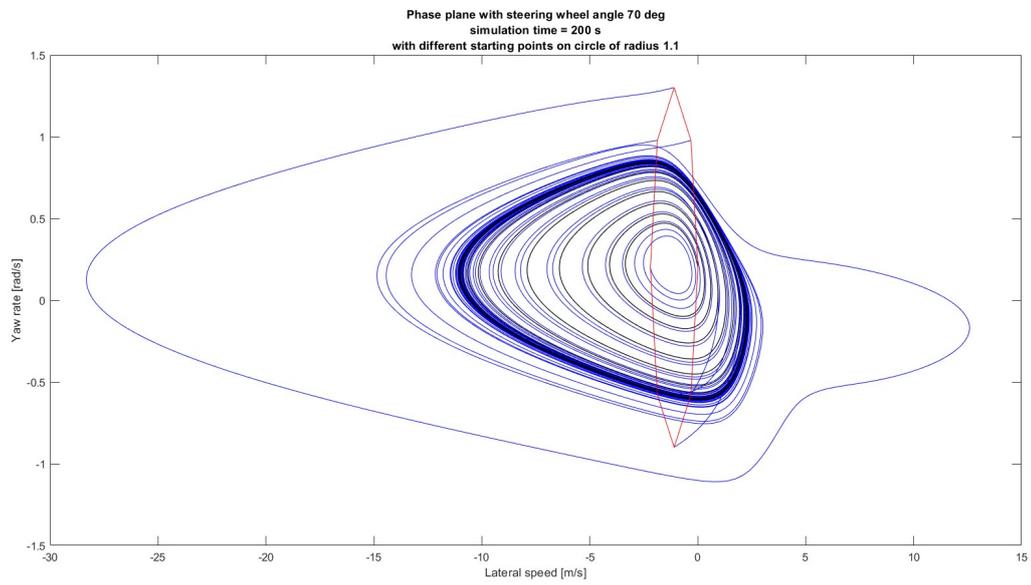


Figure 5.5. Phase plane for $\delta_{strwhl} = 70^\circ$ and $t = 200s$

6. Conclusion

Utilizing phase plane analysis and Lyapunov analysis techniques, we gained invaluable insights into the behavior of the system under various conditions. One of the significant achievements of this study was the derivation and analysis of the lateral bicycle model, which provided a foundation for understanding the lateral dynamics of vehicles with greater precision. Subsequently, we expanded upon this model by incorporating tire relaxation effects, recognizing their crucial role in real-world vehicle dynamics. Using Simulink, we tested our model with different steering wheel angles. While the vehicle may exhibit instability for certain simulation duration, further analysis uncovered the presence of stable limit cycles. By blending theoretical analysis with practical simulations, we've gained insights into the intricate dynamics of vehicular motion.

Literature

- [1] Slotine J., Li W. - Applied Nonlinear Control - Pearson, New Jersey, Prentice Hall, 1991
- [2] Vukić Z., Kuljača Lj., Đonlagić D., Tešnjak S. - Nonlinear Control Systems - New York, Marcel Dekker Inc., 2003
- [3] https://www.researchgate.net/figure/Dynamic-bicycle-model_fig1_357040239
- [4] Rajesh Rajamani - Vehicle Dynamics and Control - Springer, New York, Dordrecht, Heidelberg, London, 2012
- [5] H.B. Pacejka, J.P. Maurice - Relaxation Length Behaviour of Tyres, Vehicle System Dynamics: International Journal of Vehicle Mechanics and Mobility, 7:S1, 339-342, 1997
- [6] https://www.researchgate.net/figure/Longitudinal-vehicle-dynamics-schematic_fig1_283965703
- [7] https://en.wikipedia.org/wiki/Phase_portrait#/media/File:Van_der_pols_equation_phase_portrait.svg

7. Abstract

7.1. In Croatian

Analiza nelinearnog modela vozila pomoću teorije stabilnosti po Ljapunovu. Prvo je dana teorija nelinearnih sustava. Bočni model vozila opisan je korištenjem drugog Newtonovog zakona, kutovi klizanja izvedeni su iz geometrije prednjih i stražnjih kotača, a duljina relaksacije gume uvedena je u model. Trajektorije stanja su prikazane u faznoj ravnini pomoću simulacije modela u Simulinku s obzirom da se modelira bočno gibanje, pretpostavlja se da je longitudinalna brzina konstanta. Uspoređeni su rezultati simulacije stabilnog i nestabilnog modela. Prikazan je utjecaj relaksacije gume na bočnu stabilnost. Ljapunovljeva funkcija se koristi u algoritmu za pronalaženje ulaza za koji stabilni sustav postaje nestabilan.

7.2. In English

Nonlinear analysis of vehicle model using Lyapunov stability theory. First, theory of nonlinear systems is given. Lateral vehicle model is described using Newton's second law, slip angles are derived from geometry of front and rear wheels and tyre relaxation length is introduced into model. Trajectories are shown in phase plane plot based on data from simulation of vehicle model in Simulink. Since lateral motion is modeled, longitudinal speed is assumed constant. Results from simulation of stable and unstable model are compared. Impact of tyre relaxation on lateral stability is shown. Lyapunov's function is used in an algorithm to find input for which stable system becomes unstable.

8. Keywords

8.1. In Croatian

model automobila; nelinearni sustav; analiza nelinearnog sustava; Ljapunov teorem stabilnosti; model bicikle; fazna ravina; prikaz trajektorija

8.2. In English

vehicle model; non-linear system; analysis of nonlinear system; Lyapun's stability theorem; bicycle model; phase plane; trajectory plot